Nabitá relativistická částice v elektromagnetickém poli

pro volnou částici bez elmag. pole $L = -m_0c^2\sqrt{1-\frac{v^2}{c^2}}$ kde $v^2 = \sum \dot{x}_j^2$

Taylorův rozvoj $-m_0c^2\left(1-\frac{1}{2}\frac{v^2}{c^2}-\frac{1}{8}\frac{v^4}{c^4}-\dots\right) = \overbrace{-m_0c^2}^{\text{konst.}} + \overbrace{\frac{1}{2}m_0v^2}^{T} + \overbrace{\frac{1}{8}m_0\frac{v^4}{c^2}+\dots}^{\text{moc malé}}$

v elmag. poli $L = -m_0c^2\sqrt{1-\frac{v^2}{c^2}} - e(\varphi - \vec{v}\cdot\vec{A})$

$$p_i = \frac{\partial L}{\partial v_i} = -m_0c^2\frac{-\frac{2v_j\delta_{ij}}{c^2}}{2\sqrt{1-\frac{v^2}{c^2}}} + e\delta_{ij}A_j = \frac{m_0v_i}{\sqrt{1-\frac{v^2}{c^2}}} + eA_i$$

$$p_i - eA_i = \frac{m_0v_i}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (p_i - eA_i)^2 = \frac{m_0^2v_i^2}{1-\frac{v^2}{c^2}} \qquad (\vec{p}-e\vec{A})^2 = \frac{m_0^2c^2v^2}{c^2-v^2} \qquad v^2 = \frac{c^2(\vec{p}-e\vec{A})^2}{(\vec{p}-e\vec{A})^2+m_0^2c^2}$$

$$E = \frac{\partial L}{\partial v_i}v_i - L = \frac{m_0v_i^2}{\sqrt{1-\frac{v^2}{c^2}}} + eA_iv_i - \left(-m_0c^2\sqrt{1-\frac{v^2}{c^2}} - e(\varphi - \vec{v}\cdot\vec{A})\right) = \frac{m_0v^2+m_0c^2-m_0v^2}{\sqrt{1-\frac{v^2}{c^2}}} + e\varphi = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} + e\varphi$$

$$H = \frac{m_0c^2}{\sqrt{1-\frac{1}{c^2}\frac{c^2(\vec{p}-e\vec{A})^2}{(\vec{p}-e\vec{A})^2+m_0^2c^2}}} + e\varphi = \frac{m_0c^2}{\sqrt{\frac{m_0^2c^2}{(\vec{p}-e\vec{A})^2+m_0^2c^2}}} + e\varphi = c\sqrt{(\vec{p}-e\vec{A})^2 + m_0^2c^2} + e\varphi$$

$\vec{p}$ … kanonická hybnost, m_0 … klidová hmotnost

Věta o viriálu

Střední časová hodnota funkce $f:\mathbb{R}\to\mathbb{R}$, $f=f(t)$ $\qquad \langle f\rangle = \lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^\tau f(t)\,dt$

Věta: Pokud $\exists F:\mathbb{R}\to\mathbb{R}$, $f(t) = \frac{dF(t)}{dt}$, F omezená $(\exists K\in\mathbb{R}, |F(t)|\le K, \forall t)$ pak $\langle f\rangle = \langle\dot{F}\rangle = 0$.

Důkaz: $|\langle f\rangle| = \lim_{\tau\to+\infty}\left|\frac{1}{\tau}(F(\tau)-F(0))\right| \le \lim_{\tau\to+\infty}\left|\frac{2K}{\tau}\right| = 0$

Věta o viriálu v Hamiltonově formalizmu:

Jsou-li $q_i(t)$ a $p_i(t)$ $\forall i\in\hat{s}$ omezené funkce pak $\left\langle q_i\frac{\partial H}{\partial q_i}\right\rangle = \left\langle p_i\frac{\partial H}{\partial p_i}\right\rangle$ (pokud existují).

Značení: pro funkci $Z = Z(\vec{x}_1,\dots,\vec{x}_N,\vec{v}_1,\dots,\vec{v}_m,t)$, $Z:\mathbb{R}^{6m+1}\to\mathbb{R}$ a funkce $\vec{x}_\alpha = \vec{x}_\alpha(t)$ $\forall\alpha\in\hat{N}$

označíme $\tilde{Z} = \tilde{Z}(t) = Z(\vec{x}_1(t),\dots,\vec{x}_m(t),\dot{\vec{x}}_1(t),\dots,\dot{\vec{x}}_m(t),t)$, $\tilde{Z}:\mathbb{R}\to\mathbb{R}$ složenou funkci

(1870 Rudolf Clausius) – vztah pro střední časovou hodnotu kinetické energie

Věta o viriálu: Označme $G = \sum_{\alpha=1}^m \vec{p}_\alpha\cdot\vec{x}_\alpha$ a $T = \frac{1}{2}\sum_{\alpha=1}^m m_\alpha\vec{v}_\alpha^2$ pak pro libovolné řešení $\vec{x}_\alpha = \vec{x}_\alpha(t)$ Newtonových pohybových rovnic $m_\alpha\ddot{\vec{x}}_\alpha = \vec{F}_\alpha$ $\forall\alpha\in\hat{N}$ platí $\left\langle\frac{d\tilde{G}}{dt}\right\rangle = 0 \Rightarrow \langle\tilde{T}\rangle = \underbrace{-\frac{1}{2}\left\langle\sum_{\alpha=1}^N\tilde{\vec{F}}_\alpha\cdot\vec{x}_\alpha\right\rangle}_{\text{viriál}}$

Důkaz: $2\tilde{T} = \sum_\alpha m_\alpha\dot{\vec{x}}_\alpha^2 = \left(\sum_\alpha m_\alpha\dot{\vec{x}}_\alpha\cdot\vec{x}_\alpha\right)^{\bullet} - \sum_\alpha m_\alpha\ddot{\vec{x}}_\alpha\cdot\vec{x}_\alpha = \dot{\tilde{G}} - \sum_\alpha\tilde{\vec{F}}_\alpha\cdot\vec{x}_\alpha$

Jsou-li navíc síly $\vec{F}_\alpha$ potenciální tj. $\vec{F}_\alpha = -\nabla_\alpha U = -\frac{\partial U}{\partial\vec{x}_\alpha}$ a potenciál $U = U(\vec{x}_1,\dots,\vec{x}_N,t)$ je homogenní funkce stupně k v proměnných $\vec{x}_1,\dots,\vec{x}_N$ pak $\langle\dot{\tilde{G}}\rangle = 0 \Rightarrow \langle\tilde{T}\rangle = \frac{k}{2}\langle\tilde{U}\rangle$

Důkaz: $\sum_\alpha\tilde{\vec{F}}_\alpha\cdot\vec{x}_\alpha = \sum_{\alpha=1}^N\sum_{i=1}^3\tilde{F}_{\alpha_i}x_{\alpha_i} = -\sum_{\alpha=1}^N\sum_{i=1}^3\frac{\partial U}{\partial x_{\alpha_i}}x_{\alpha_i} = -kU$

Označíme-li $E = T+U$ pak $\langle\tilde{E}\rangle = \langle\tilde{T}\rangle + \langle\tilde{U}\rangle = \left(1+\frac{k}{2}\right)\langle\tilde{U}\rangle$ a platí $\langle\tilde{U}\rangle = \frac{2}{k+2}\langle\tilde{E}\rangle$

$\langle\tilde{T}\rangle = \frac{k}{k+2}\langle\tilde{E}\rangle$

Je-li navíc $\frac{\partial U}{\partial t} = 0$ pak $\tilde{E} = $ konst. je celková energie a $\langle\tilde{E}\rangle = E$