Hamilton–Jacobiho rovnice

hledáme $F_2(\vec q,\vec P,t)$ tak, že $K = H' = \underbrace{H + \frac{\partial F_2}{\partial t} = 0}$ $\Rightarrow$ $\boxed{\begin{aligned}\dot Q_i &= \frac{\partial K}{\partial P_i} = 0 \Rightarrow Q_i = \text{konst}\\ \dot P_i &= -\frac{\partial K}{\partial Q_i} = 0 \Rightarrow \underline{P_i = \text{konst.}}\end{aligned}}$ $\forall i \in \hat s$

$\downarrow$

$S(\vec q, t, \vec P)$ 2s+1 proměnných

HJR $\boxed{H\left(\vec q, \left(\frac{\partial S}{\partial \vec q}\right), t\right) + \frac{\partial S}{\partial t} = 0}$ $\leftrightarrow$ $s + \cancel{1}$ konst. $\quad \tilde S = S + \cancel{\text{konst}}$

$\downarrow$ $\vec P$

24) Vyřešte Hamilton–Jacobiho rovnici pro bezsilový volný hmotný bod. Ukažte, že $S_1(\vec q,t,\vec Q) = \frac{m}{2t}\sum_{i=1}^{3}(q_i - Q_i)^2$ a $S_2(\vec q,t,\vec P) = \sum_{i=1}^{3}\left(-\frac{P_i^2}{2m}t + P_i q_i\right)$ jsou její úplné integrály.

$H = \frac{1}{2m}p_i^2 \leftarrow p_i = \frac{\partial S}{\partial q_i}$ $\qquad H + \frac{\partial S}{\partial t} = 0 \longrightarrow \boxed{\underbrace{\sum_i \frac{1}{2m}\left(\frac{\partial S}{\partial q_i}\right)^2}_{H} + \frac{\partial S}{\partial t} = 0}$

Řešení – separací proměnných $S(\vec q, t) = S_t(t) + S_0(\vec q)$ $\qquad \frac{\partial H}{\partial t} = 0 \Rightarrow H = E$ konst.

$\sum \frac{1}{2m}\left(\frac{\partial S_0}{\partial q_i}\right)^2 + \frac{dS_t}{dt} = 0 \quad \rightarrow \quad \underbrace{\sum_i \frac{1}{2m}\left(\frac{\partial S_0}{\partial q_i}\right)^2}_{\text{fce od } \vec q} \overset{②}{=} \text{konst.} \overset{①}{=} \underbrace{-\frac{dS_t(t)}{dt}}_{\text{fce od } t}$

E

① $\frac{dS_t(t)}{dt} = -E \Rightarrow S_t(t) = \int -E\,dt + C$

$\underline{S_t(t) = -Et} + \cancel{C}$, $C = 0$

② $\sum_i \frac{1}{2m}\left(\frac{\partial S_0}{\partial q_i}\right)^2 = E \quad /2m$ Separace proměnných $S_0(\vec q) = S_1(q_1) + S_2(q_2) + S_3(q_3)$

$\underbrace{\left(\frac{dS_1(q_1)}{dq_1}\right)^2}_{f_1 \text{ fce jen } q_1} + \underbrace{\left(\frac{dS_2(q_2)}{dq_2}\right)^2}_{\text{fce } q_2} + \underbrace{\left(\frac{dS_3(q_3)}{dq_3}\right)^2}_{\text{fce } q_3} = \underbrace{2mE}_{\text{konst.}}$ $\qquad \left.\begin{aligned}q_2 &= \text{konst}\\ q_3 &= \text{konst}\end{aligned}\right\} \Rightarrow f_1(q_1) = \text{konst.}\ \forall q_1$

$\overset{\boxed{1}}{=} P_1^2 + \overset{\boxed{2}}{=} P_2^2 + \overset{\boxed{3}}{=} P_3^2 = 2mE \quad \longrightarrow \quad E = \sum_i \frac{P_i^2}{2m}$

$\boxed{1}$ $\frac{dS_1}{dq_1} = P_1 \Rightarrow S_1(q_1) = \int P_1\,dq_1 + C_1$ $\qquad \boxed{2}$ $\frac{dS_2}{dq_2} = P_2 \Rightarrow \underline{S_2(q_2) = P_2 q_2} + \cancel{C}$ $\qquad \boxed{3}$ $\frac{dS_3}{dq_3} = P_3 \Rightarrow \underline{S_3(q_3) = P_3 q_3} + \cancel{C_3}$

$\underline{S_1(q_1) = P_1 q_1} + \cancel{C_1}$

$\underline{S(\vec q, t) = S_t(t) + S_0(\vec q)} = -Et + P_1 q_1 + P_2 q_2 + P_3 q_3$ $\qquad \boxed{S(\vec q, t, \vec P) = -\sum_i \frac{P_i^2}{2m}t + \sum_i P_i q_i} = S_②$

Pozn. $P_1 = E \Rightarrow S(\vec q, t, \vec P) = -P_1 t + \sqrt{(P_1 - P_2^2 - P_3^2)}\,q_1 + P_2 q_2 + P_3 q_3$

$S_① = S_② - P_i Q_i = -\frac{P_i^2}{2m}t + P_i q_i - P_i Q_i \overset{i \to j}{=} -\frac{(q_j - Q_j)^2}{2m\,t^2}m^2 t + \frac{(q_j - Q_j)}{t}m(q_j - Q_j) = \underline{\underline{\frac{m}{2t}(q_j - Q_j)^2}}$

Legendreova duální Tr.

$Q_j = \frac{\partial S_②}{\partial P_j} = -\frac{1}{2m}2P_i\delta_{ij} + q_i\delta_{ij} = -\frac{1}{m}P_j + q_j \Rightarrow \underline{P_j = \frac{q_j - Q_j}{t}m}$

HJR $\frac{1}{2m}\left(\frac{\partial S_①}{\partial q_i}\right)^2 - \frac{\partial S_①}{\partial t} \overset{?}{=} 0$ DCV.

25) Zkonstruujte hlavní funkci Hamiltonovu $S_①(\vec q, t, \vec Q) = \frac{m}{2t}\sum_{i=1}^{3}(q_i - Q_i)^2$ integrací Lagrangeovy funkce volného bezsilového hm. bodu po jeho skutečné trajektorii. $\qquad L = \frac{1}{2}m\dot q_j^2 \quad q_j(t) = v_j t + Q_j$

$\left.\frac{dS_①}{dt}\right|_{\substack{\text{podél}\\ \text{trajektorie}\\ \text{(H.R.)}}} = \underbrace{\frac{\partial S}{\partial q_i}}_{p_i}\dot q_i + \underbrace{\frac{\partial S}{\partial t}}_{-H} + \underbrace{\frac{\partial S}{\partial Q_i}\dot Q_i}_{0} = p_i\dot q_i - H = L$ Legendreova Tr.

$S_①(\vec q(t), t, \vec Q) = \int_{t_0(\vec Q)}^{t(\vec q(t))} L\,d\tilde t$ $\qquad S = \int_{t_0}^{t}\frac{1}{2}m\dot q_j^2\,dt = \int_{t_0}^{t}\frac{1}{2}m v_j^2\,dt =$

t_0 čas kdy $q_j = Q_j$ $\qquad = \left[\frac{1}{2}m v_j^2 t\right]_{t_0}^{t} = \frac{1}{2}m v_j^2 t =$

$\underline{t_0 = 0}$ $\qquad t_0 = 0$

$v_j = \frac{q_j - Q_j}{t}$ $\qquad = \frac{1}{2}\frac{m(q_j - Q_j)^2}{t}$ //

26) Najděte kanonické transformace určené vytvořujícími funkcemi $\boxed{S_1(\vec q,t,\vec Q)=\frac{m}{2t}\sum_{i=1}^{3}(q_i-Q_i)^2}$ a $S_2(\vec q,t,\vec P)=\sum_{i=1}^{3}\left(-\frac{P_i^2}{2m}t+P_iq_i\right)$ Jaký tvar mají příslušné vlnoplochy S=konst. v konfiguračním prostoru?

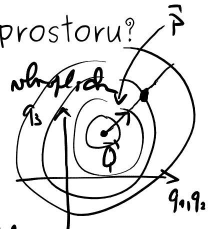
$p_j=\frac{\partial S_1}{\partial q_j}=\frac{m}{2t}(q_i-Q_i)2\cdot\delta_{ij}=\frac{m}{t}(q_j-Q_j)\Rightarrow \boxed{p_j=P_j}$

$\boxed{P_j=-\frac{\partial S_1}{\partial Q_j}}=-\frac{m}{2t}(q_i-Q_i)2(-\delta_{ij})=\frac{m}{t}(q_j-Q_j)\Rightarrow \boxed{q_j=\frac{P_j}{m}t+Q_j}$

$p_j=\frac{\partial S_2}{\partial q_j}=P_i\delta_{ij}=P_j \Rightarrow \boxed{p_j=P_j}$

$Q_j=\frac{\partial S_2}{\partial P_j}=-\frac{t}{2m}2P_i\delta_{ij}+q_i\delta_{ij}=-\frac{t}{m}P_j+q_j\Rightarrow \boxed{q_j=\frac{t}{m}P_j+Q_j}$

pro dané t $\quad S_1(\vec q,t,\vec Q)=\text{konst}$ → vlnoplocha



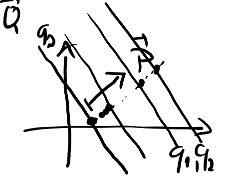
$\frac{m}{2t}(q_i-Q_i)^2=\text{konst.}=S_1$

$\boxed{(q_i-Q_i)^2=\frac{S_1\cdot 2t}{m}}$ sféry s počátkem v $\vec Q$

$S_2(\vec q,t,\vec P)=\text{konst.}$

$\boxed{P_iq_i=S_2+\frac{P_i^2}{2m}t}$

Rovina s normálou $\vec P$



27) Napište a řešte Hamilton–Jacobiho rovnici pro lineární harmonický oscilátor $H=\frac{p^2}{2m}+\frac{k}{2}q^2$

$p=\frac{\partial S}{\partial q}$ HJR $\underbrace{\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{k}{2}q^2}_{H=E}+\frac{\partial S}{\partial t}=0 \qquad \frac{\partial H}{\partial t}=0\Rightarrow H=E$ I.P.

$S(q,t)=S_0(q)+S_t(t)$

$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{k}{2}q^2=E \qquad \frac{\partial S}{\partial t}=-E \qquad \frac{dS_t}{dt}=-E\Rightarrow S_t(t)=-Et+\cancel{C}$

$\underbrace{\frac{1}{2m}\left(\frac{dS_0}{dq}\right)^2+\frac{k}{2}q^2}_{\text{fce od } q}=E \Rightarrow \left(\frac{dS_0}{dq}\right)^2=2mE-kmq^2=2mE\left(1-\frac{k}{2E}q^2\right) \quad /\sqrt{\ } \;/\,(+) \qquad x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\frac{dS_0}{dq}=\sqrt{2mE}\sqrt{\left(1-\frac{k}{2E}q^2\right)} \quad /\int dq \quad\to\quad S_0=\sqrt{2mE}\int\sqrt{\left(1-\frac{k}{2E}q^2\right)}\,dq=\left|\begin{array}{l}\sin x=\sqrt{\frac{k}{2E}}q\\ \cos x\,dx=\sqrt{\frac{k}{2E}}dq\end{array}\right|=$

$=\sqrt{2mE}\int\underbrace{\sqrt{1-\sin^2x}}_{\cos x}\sqrt{\frac{2E}{k}}\cos x\,dx=\sqrt{\frac{m}{k}}2E\int\underbrace{\cos^2x}_{\frac{1+\cos(2x)}{2}}dx=\sqrt{\frac{m}{k}}2E\left(\frac{x}{2}+\frac{\sin(2x)}{4}\right)$

$S_0(q)=\underline{E\sqrt{\frac{m}{k}}}\left(\arcsin\left(\sqrt{\frac{k}{2E}}q\right)+\overbrace{\sqrt{\frac{k}{2E}}q\sqrt{1-\frac{k}{2E}q^2}}^{\sin x\cos x}\right)$

Volba $P=E \qquad S(q,t,P)=-Pt+P\sqrt{\frac{m}{k}}\left(\arcsin\left(\sqrt{\frac{k}{2P}}q\right)+\sqrt{\frac{k}{2P}}q\sqrt{1-\frac{k}{2P}q^2}\right)$

Jiná volba $P=\sqrt{2mE}\to E=\frac{P^2}{2m} \qquad \boxed{S(q,t,P)=-\frac{P^2}{2m}t+\frac{P^2}{2\sqrt{mk}}\arcsin\left(\sqrt{km}\frac{q}{P}\right)+\frac{1}{2}q\sqrt{P^2-kmq^2}}$

kanonická Tr.

$p=\frac{\partial S}{\partial q}=\frac{P^2}{2\cancel{\sqrt{km}}}\frac{\cancel{\sqrt{km}}\frac{1}{P}}{\sqrt{1-km\left(\frac{q}{P}\right)^2}}+\frac{1}{2}\sqrt{P^2-kmq^2}+\frac{1}{2}q\frac{-km\cancel{2}q}{\cancel{2}\sqrt{P^2-kmq^2}}=\frac{P^2+P^2-kmq^2-kmq^2}{2\sqrt{P^2-kmq^2}}=\boxed{\sqrt{P^2-kmq^2}}$

$Q=\frac{\partial S}{\partial P}=-\frac{2P}{2m}t+\frac{2P}{2\sqrt{km}}\arcsin\left(\sqrt{km}\frac{q}{P}\right)+\cancel{\frac{P^2}{2\sqrt{km}}\frac{\sqrt{km}q\left(-\frac{1}{P^2}\right)}{\sqrt{1-km\frac{q^2}{P^2}}}}+\cancel{\frac{1}{2}q\frac{2P}{2\sqrt{P^2-kmq^2}}}=\boxed{-\frac{P}{m}t+\frac{P}{\sqrt{km}}\arcsin\left(\sqrt{km}\frac{q}{P}\right)}$

$q(t,Q,P)=\sin\left(\frac{\sqrt{km}}{P}\left(Q+\frac{P}{m}t\right)\right)\frac{P}{\sqrt{km}}=\boxed{\frac{P}{\sqrt{km}}\sin\left(\sqrt{\frac{k}{m}}t+\sqrt{km}\frac{Q}{P}\right)}$

$\boxed{p(t,Q,P)=P\cos\left(\sqrt{\frac{k}{m}}t+\sqrt{km}\frac{Q}{P}\right)}$ Fázová trajektorie