Lagrangeova a Hamiltonova fce. pro částici s nábojem q v elmag. poli s potenciály $\varphi=\varphi(\vec{x},t)$ $\vec{A}=\vec{A}(\vec{x},t)$

nerelativisticky

$$L=\frac{1}{2}m\vec{v}^2-q(\varphi-\vec{v}\cdot\vec{A})$$

$$H=\frac{1}{2m}(\vec{p}-q\vec{A})^2+q\varphi$$

relativisticky

$$L=-m_0c^2\sqrt{1-\frac{v^2}{c^2}}-q(\varphi-\vec{v}\cdot\vec{A})$$

$L(\vec{x},\vec{v},t)$ $\quad \vec{v}=\dot{\vec{x}}=\frac{d\vec{x}}{dt}$

duální Legendreova Transf.

$$p_i=\frac{\partial L}{\partial v_i}=\frac{\partial L}{\partial \dot{x}_i}$$

$$H=c\sqrt{m_0^2c^2+(\vec{p}-q\vec{A})^2}+q\varphi$$

Jinak:

$$E^2=m_0^2c^4+c^2p^2 \Rightarrow E=c\sqrt{m_0^2c^2+p^2}$$ ← pro volnou bezsilovou částici

$E \updownarrow H$, $\quad p \updownarrow$ obecná hybnost $\quad \xleftrightarrow{\text{Legendreova Tr.}} \quad L=-m_0c^2\sqrt{1-\frac{v^2}{c^2}}$

53) Ukažte, že při pohybu nabité částice (m_0,q) v magnetickém poli určeném vektorovým potenciálem $\vec{A}=(0,0,A(x,y))$ se zachovává veličina $\frac{m_0v_z}{\sqrt{1-\frac{v^2}{c^2}}}+qA(x,y)$ $\quad\rightarrow \varphi=0 \quad (\vec{E}=0)$

$$L=-m_0c^2\sqrt{1-\frac{v^2}{c^2}}-q(\varphi-\vec{v}\cdot\vec{A})=-m_0c^2\sqrt{1-\frac{1}{c^2}(\underbrace{v_x^2+v_y^2+v_z^2}_{\dot{x}^2+\dot{y}^2+\dot{z}^2})}+q\underbrace{v_z}_{\dot{z}}A(x,y)$$

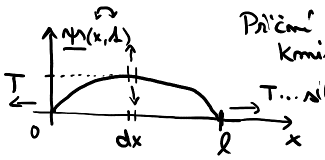
$\frac{\partial L}{\partial z}=0 \Rightarrow z$ cyklická a

tedy

$$p_z=\underbrace{\frac{\partial L}{\partial \dot{z}}}_{\text{je I.P.}}=\frac{\partial L}{\partial v_z}=-m_0c^2\frac{\left(-\frac{1}{c^2}2v_z\right)}{2\sqrt{1-\frac{v^2}{c^2}}}+qA(x,y)=\underbrace{\frac{m_0v_z}{\sqrt{1-\frac{v^2}{c^2}}}}+qA(x,y)=\underline{\underline{\text{konst.}}}$$

$$\frac{\partial L}{\partial t}=0 \Rightarrow E=\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i-L=\frac{m_0\overbrace{(v_x^2+v_y^2+v_z^2)}^{v^2}}{\sqrt{1-\frac{v^2}{c^2}}}+\cancel{qv_zA(x,y)}-\left(-m_0c^2\sqrt{1-\frac{v^2}{c^2}}+\cancel{qv_zA(x,y)}\right)=\frac{m_0v^2+m_0c^2\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}=\underline{\underline{\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}}}$$

Př Pole - struna (vocf) Hustota kinetické energie $\mathcal{T}=\frac{1}{2}\rho\left(\frac{\partial\psi}{\partial t}\right)^2$ $\quad$ Hustota Lagrangeovy funkce



Příčné kmity

T... síla napínající strunu

ρ... hustota (lineární) struny

Potenciální energie $\mathcal{U}=\frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2$

$$\boxed{\mathcal{L}=\mathcal{T}-\mathcal{U}=\frac{1}{2}\rho\psi_t^2-\frac{1}{2}T\psi_x^2}$$

$\psi_t=\frac{\partial\psi}{\partial t} \quad \psi_x=\frac{\partial\psi}{\partial x}$

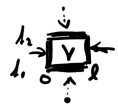
"$L=\int_0^l \mathcal{L}\,dx$" $\quad$ pole popsáno jednou funkcí $\psi=\psi(x,t)$

$T=\int_0^l \mathcal{T}\,dx \quad U=\int_0^l \mathcal{U}\,dx$

Pevné konce $\psi(0,t)=0=\psi(l,t) \quad \delta\psi(0,t)=0=\delta\psi(l,t)$

Počáteční podmínky $\psi(x,t_1)=g_1(x) \quad \psi(x,t_2)=g_2(x)\Rightarrow\delta\psi(x,t_{1,2})=0$ $\Big\}\ \delta\psi|_{\partial V}=0$

$$S=\int_{t_1}^{t_2}L\,dt=\int_{t_1}^{t_2}\int_0^l\mathcal{L}\,dx\,dt=\int_V\mathcal{L}\,dV$$



Hamiltonův princip pro pole

Skutečný časový vývoj soustavy polí q_α se děje s takovou závislostí q_α na x^μ pro kterou akce S nabývá stacionární hodnoty vzhledem k variacím $\delta q_\alpha(x^\mu)$ splňujícím podmínku pevných konců $\delta q_\alpha(x^\mu)\big|_{\partial V^*}=0$.

akce $S[q_\alpha(x^\mu)]=\frac{1}{c}\int_{V^*}\mathcal{L}(q_\alpha,q_{\alpha,\mu},x^\mu)\,dV^*$ $\quad q_{\alpha,\mu}=\frac{\partial q_\alpha}{\partial x^\mu} \quad \mu=0,1,2,3$

Hustota Lagr. fce.

souřadnice - proměnné pole

$\alpha=\dots$ podle typu a počtu polí

$V^*\dots$ objem oblasti v $\mathbb{R}^4$ $\quad dV^*=dx^0dx^1dx^2dx^3=c\,dt\,\underbrace{dx\,dy\,dz}_{dV}$

Speciálně pro $V^*=\langle ct_1,ct_2\rangle\times V$, $V\subset\mathbb{R}^3$

$$S=\frac{1}{c}\int_{V^*}\mathcal{L}\,dV^*=\frac{1}{c}\int_{t_1}^{t_2}\int_V\mathcal{L}\,c\,dV\,dt=\int_{t_1}^{t_2}\underbrace{\left(\int_V\mathcal{L}\,dV\right)}_{\text{"}L\text{"}}dt$$

$$0=\delta S=\frac{1}{c}\int_{V^*}\delta\mathcal{L}\,dV^*=\frac{1}{c}\int_{V^*}\left[\frac{\partial\mathcal{L}}{\partial q_\alpha}\delta q_\alpha+\frac{\partial\mathcal{L}}{\partial q_{\alpha,\mu}}\delta q_{\alpha,\mu}\right]dV^*=\frac{1}{c}\int_{V^*}\left[\frac{\partial\mathcal{L}}{\partial q_\alpha}\delta q_\alpha+\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial q_{\alpha,\mu}}\delta q_\alpha\right)-\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial q_{\alpha,\mu}}\right)\cdot\delta q_\alpha\right]dV^*=$$

$$=\frac{1}{c}\int_{V^*}\left[\frac{\partial\mathcal{L}}{\partial q_\alpha}-\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial q_{\alpha,\mu}}\right)\right]\delta q_\alpha\,dV^*+\frac{1}{c}\int_{V^*}\underbrace{\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial q_{\alpha,\mu}}\delta q_\alpha\right)}_{\mu\ \text{čtyřdivergence}}dV^*=\frac{1}{c}\int_{V^*}\left[\frac{\partial\mathcal{L}}{\partial q_\alpha}-\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial q_{\alpha,\mu}}\right)\right]\underbrace{\delta q_\alpha}_{\text{Nezávislé}}dV^*+\cancel{\frac{1}{c}\oint_{\partial V^*}\frac{\partial\mathcal{L}}{\partial q_{\alpha,\mu}}\delta q_\alpha\,df_\mu}$$

Pevné konce $\rightarrow 0$

$$\int_V d\omega = \int_{\partial V} \omega$$

<u>Lagrangeovy rovnice pro pole</u>

$$\boxed{\sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial q_{\alpha,\mu}}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0 \quad \forall \alpha}$$

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \ \forall i \in \hat{n}\right)$$

Pozn.:

$$\frac{\partial}{\partial x^0} = \frac{\partial}{\partial (ct)} = \frac{\partial t}{\partial (ct)} \cdot \frac{\partial}{\partial t} = \frac{\partial\left(\frac{1}{c} ct\right)}{\partial (ct)} \frac{\partial}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}$$

$$q_{\alpha,0} = \frac{\partial q_\alpha}{\partial x^0} = \frac{1}{c}\frac{\partial q_\alpha}{\partial t} = \frac{1}{c} q_{\alpha,t} \qquad \frac{\partial \mathcal{L}}{\partial q_{\alpha,0}} = \frac{\partial \mathcal{L}}{\partial\left(\frac{1}{c} q_{\alpha,t}\right)} = c\frac{\partial \mathcal{L}}{\partial q_{\alpha,t}}$$

$$\frac{\partial}{\partial x^0}\left(\frac{\partial \mathcal{L}}{\partial q_{\alpha,0}}\right) = \frac{1}{c}\left(\frac{\partial}{\partial t}\left(c\frac{\partial \mathcal{L}}{\partial q_{\alpha,t}}\right)\right) = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial q_{\alpha,t}}\right)$$

54) Odvoďte pohybové rovnice polí v dvourozměrném prostoročase (t,x) z Lagrangeových hustot

a) $\mathcal{L} = \frac{1}{2}(\varphi_t^2 - \varphi_x^2) + \frac{1}{2}\mu^2\varphi^2 - \frac{1}{4}\lambda\varphi^4$ $\quad \mu^2, \lambda > 0$ jsou konst.

$\mathcal{L} = \mathcal{L}(\varphi, \varphi_t, \varphi_x)$

Pole - jedno $\varphi = \varphi(t,x)$, $q_\alpha \leftrightarrow \varphi$, $\varphi_t = \frac{\partial \varphi}{\partial t}$, $\varphi_x = \frac{\partial \varphi}{\partial x}$

$$\frac{\partial}{\partial x^\mu}\boxed{\frac{\partial \mathcal{L}}{\partial q_{\alpha,\mu}}} \qquad \boxed{\frac{\partial \mathcal{L}}{\partial q_\alpha}}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_t} = \frac{1}{2}\cdot 2\varphi_t = \varphi_t \qquad \frac{\partial \mathcal{L}}{\partial \varphi_x} = \frac{1}{2}(-2)\varphi_x = -\varphi_x$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{1}{2}\mu^2 2\varphi - \frac{1}{4}\lambda 4\varphi^3 = \mu^2\varphi - \lambda\varphi^3$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \varphi_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi_x}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\varphi_{tt} + (-\varphi_{xx}) - (\mu^2\varphi - \lambda\varphi^3) = 0$$

$$\boxed{\varphi_{tt} - \varphi_{xx} - \mu^2\varphi + \lambda\varphi^3 = 0}$$

b) sinus-Gordon $\mathcal{L} = \frac{1}{2}(\varphi_t^2 - \varphi_x^2) + (\cos\varphi - 1)$ DCV.

c) Korteweg - de Vries $\mathcal{L} = \frac{1}{2}\Theta_x\Theta_t + \frac{\alpha}{6}\Theta_x^3 + \Theta_x\Psi_x + \frac{1}{2}\Psi^2$ kde α = konst.

dvě pole $\Theta = \Theta(t,x) \leftrightarrow q_1$, $\Psi = \Psi(t,x) \leftrightarrow q_2$, $\alpha = 1,2$

$\mathcal{L}(\Psi, \Psi_x, \Theta_x, \Theta_t)$

$$\Psi \qquad 0 = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \Psi_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \Psi_x}\right) - \frac{\partial \mathcal{L}}{\partial \Psi} = \frac{\partial}{\partial t}(0) + \frac{\partial}{\partial x}(\Theta_x) - (\Psi) = \Theta_{xx} - \Psi$$

$$\Theta_{xt} = \Theta_{tx}$$

$$\Theta \qquad 0 = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \Theta_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \Theta_x}\right) - \frac{\partial \mathcal{L}}{\partial \Theta} = \frac{\partial}{\partial t}\left(\frac{1}{2}\Theta_x\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\Theta_t + \frac{\alpha}{2}\Theta_x^2 + \Psi_x\right) - 0 = \frac{1}{2}\Theta_{xt} + \frac{1}{2}\Theta_{tx} + \frac{\alpha}{2}2\Theta_x\Theta_{xx} + \Psi_{xx}$$

<u>Substituce</u> $\boxed{\Theta_x = \varphi}$ → <u>KdV rovnice</u> $\quad \Psi = \Theta_{xx} = \varphi_x \quad 0 = \Theta_{xt} + \alpha\Theta_x\Theta_{xx} + \Psi_{xx} = \varphi_t + \alpha\varphi\varphi_x + \varphi_{xxx} = 0$

55) Odvoďte pohybovou rovnici reálného skalárního pole $\varphi = \varphi(t,\vec{x})$ jestliže $\mathcal{L} = \frac{1}{2}(\varphi_{,\mu}\varphi^{,\mu} - \varkappa^2\varphi^2)$ (Klein - Gordon) $\varkappa$ konst.

$$\varphi_{,\mu} = \frac{\partial \varphi}{\partial x^\mu} \qquad \varphi^{,\mu} = \frac{\partial \varphi}{\partial x_\mu} = g^{\mu\nu}\varphi_{,\nu}$$

$$\mathcal{L} = \frac{1}{2}\left(g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - \varkappa^2\varphi^2\right) = \frac{1}{2}\left(g^{\mu\nu}\frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} - \varkappa^2\varphi^2\right)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_{,\rho}} = \frac{1}{2}g^{\mu\nu}\left(\delta^\rho_\mu\varphi_{,\nu} + \varphi_{,\mu}\delta^\rho_\nu\right) = \frac{1}{2}g^{\rho\nu}\varphi_{,\nu} + \frac{1}{2}g^{\mu\rho}\varphi_{,\mu} = g^{\rho\nu}\varphi_{,\nu}$$

$$\frac{\partial}{\partial x^\rho}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,\rho}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$g^{\mu\rho} = g^{\rho\mu} \quad (g) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -\varkappa^2\varphi$$

$$\frac{\partial}{\partial x^\rho}\left(g^{\rho\nu}\varphi_{,\nu}\right) - (-\varkappa^2\varphi) = 0$$

$$\boxed{g^{\rho\nu}\varphi_{,\nu\rho} + \varkappa^2\varphi = 0}$$

$$\underbrace{g^{00}}_{1}\varphi_{,00} + \underbrace{g^{11}}_{-1}\varphi_{,11} + \underbrace{g^{22}}_{-1}\varphi_{,22} + \underbrace{g^{33}}_{-1}\varphi_{,33} + \varkappa^2\varphi = 0 \qquad \varphi_{,1} = \frac{\partial\varphi}{\partial x^1} = \frac{\partial\varphi}{\partial x}$$

$$\frac{1}{c^2}\varphi_{,tt} \underbrace{- \varphi_{,xx} - \varphi_{,yy} - \varphi_{,zz}}_{-\Delta\varphi} + \varkappa^2\varphi = 0 \qquad \underbrace{\frac{1}{c^2}\varphi_{,tt} - \Delta\varphi + \varkappa^2\varphi = 0}_{-\Box\varphi + \varkappa^2\varphi = 0}$$