61) Rozhodněte, zda je veličina $A^\mu A_\mu$ vytvořená z čtyřpotenciálu a) Lorentzovsky b) kalibračně invariantní?

$$A^\mu = \left(\frac{\varphi}{c}, \vec{A}\right) \qquad A_\mu = g_{\mu\nu}A^\nu = \left(\frac{\varphi}{c}, -\vec{A}\right)$$

$$\begin{pmatrix}1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}\begin{pmatrix}\frac{\varphi}{c}\\ \vec A\end{pmatrix}$$

$$\boxed{A'^\mu = \alpha^\mu{}_\nu A^\nu}$$

b) Kalibrační Invariance $\qquad \Lambda = \Lambda(\vec r, t)$

$$\tilde A^\mu = \left(\frac{\tilde\varphi}{c}, \tilde{\vec A}\right) = \left(\frac1c\left(\varphi - \frac{\partial\Lambda}{\partial t}\right), \vec A + \text{grad}\,\Lambda\right)$$

$$\tilde A^\mu \tilde A_\mu = \begin{pmatrix}\frac1c\left(\varphi-\frac{\partial\Lambda}{\partial t}\right)\\ \vec A+\text{grad}\,\Lambda\end{pmatrix}^T \cdot \begin{pmatrix}\frac1c\left(\varphi-\frac{\partial\Lambda}{\partial t}\right)\\ -\vec A-\text{grad}\,\Lambda\end{pmatrix} = \frac{\varphi^2}{c^2} - \frac{2\varphi}{c^2}\frac{\partial\Lambda}{\partial t} + \frac{1}{c^2}\left(\frac{\partial\Lambda}{\partial t}\right)^2 - \vec A^2 - 2(\text{grad}\,\Lambda)\cdot\vec A - (\text{grad}\,\Lambda)^2 =$$

$$= A^\mu A_\mu \underbrace{- \frac{2}{c^2}\varphi\frac{\partial\Lambda}{\partial t} + \frac{1}{c^2}\left(\frac{\partial\Lambda}{\partial t}\right)^2 - 2\vec A\cdot\text{grad}\,\Lambda - (\text{grad}\,\Lambda)^2}_{\neq 0}$$

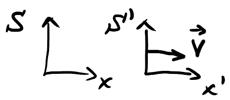
není kalibračně invariantní

a) Lorentzovsky Invariantní

$$A'^\mu A'_\mu = g'_{\mu\nu} A'^\mu A'^\nu = g'_{\mu\nu}\alpha^\mu{}_\rho A^\rho \alpha^\nu{}_\sigma A^\sigma = \underbrace{g'_{\mu\nu}\alpha^\mu{}_\rho\alpha^\nu{}_\sigma}_{g_{\rho\sigma}} A^\rho A^\sigma = g_{\rho\sigma}A^\rho A^\sigma = A^\rho A_\rho \quad ✓$$

56) Pomocí čtyř potenciálu A^μ resp. tenzoru elektromagnetického pole $F^{\mu\nu}$ odvoďte transformační vztahy pro skalární φ a vektorový potenciál $\vec A$ resp. pro $\vec E$ a $\vec B$

Lorentzova Tr.



$$x' = \gamma(x - Vt) \qquad y' = y \qquad t' = \gamma\left(t - \frac{Vx}{c^2}\right) \qquad z' = z$$

$$x'^\mu = \alpha^\mu{}_\nu x^\nu \qquad (\alpha^\mu{}_\nu) = \begin{pmatrix}\gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \qquad \beta = \frac{V}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

a) Čtyřpotenciál

$$A^\mu = \begin{pmatrix}\frac{\varphi}{c}\\ \vec A\end{pmatrix} \qquad \boxed{A'^\mu = \alpha^\mu{}_\nu A^\nu}$$

$$\begin{pmatrix}\frac{\varphi'}{c}\\ A'_x\\ A'_y\\ A'_z\end{pmatrix} = \begin{pmatrix}\gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{\varphi}{c}\\ A_x\\ A_y\\ A_z\end{pmatrix} = \begin{pmatrix}\gamma\frac{\varphi}{c} - \beta\gamma A_x\\ -\beta\gamma\frac{\varphi}{c} + \gamma A_x\\ A_y\\ A_z\end{pmatrix}$$

$$\varphi' = \gamma(\varphi - \overbrace{c\beta}^{V} A_x)$$
$$A'_x = \gamma\left(A_x - \frac{\beta}{c}\varphi\right)$$
$$A'_y = A_y$$
$$A'_z = A_z$$

$$A'^\mu(x'^\rho) = \alpha^\mu{}_\nu A^\nu(x^\sigma) = \alpha^\mu{}_\nu A^\nu\left((\alpha^{-1})^\sigma{}_\rho x'^\rho\right)$$

b) Tenzor elektromagnetického pole

$$F_{\mu\nu} := \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix}0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c}\\ -\frac{E_x}{c} & 0 & -B_z & B_y\\ -\frac{E_y}{c} & B_z & 0 & -B_x\\ -\frac{E_z}{c} & -B_y & B_x & 0\end{pmatrix}$$

$$\frac1c\frac{\partial (-A_x)}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\varphi}{c}\right)$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \text{ Kovar.} \qquad \partial^\mu = \frac{\partial}{\partial x_\mu} \text{ Kontra}$$

$$F_{\mu\nu} = -F_{\nu\mu}$$

$$F^{\mu\nu} = \underbrace{g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}}_{g\,\mathbb{F}\,g^T} = \begin{pmatrix}0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c}\\ \frac{E_x}{c} & 0 & -B_z & B_y\\ \frac{E_y}{c} & B_z & 0 & -B_x\\ \frac{E_z}{c} & -B_y & B_x & 0\end{pmatrix}$$

Transformace (Lorentzova)

$$\boxed{F'^{\mu\nu} = \alpha^\mu{}_\rho\alpha^\nu{}_\sigma F^{\rho\sigma}} = \alpha^\mu{}_\rho F^{\rho\sigma}(\alpha^T)_\sigma{}^\nu = (\alpha\,\mathbb{F}\,\alpha^T)^{\mu\nu}$$

$$\left(0 = F'^{00} = \alpha^0{}_\rho\alpha^0{}_\sigma F^{\rho\sigma} = \alpha^0{}_0\alpha^0{}_0\underbrace{F^{00}}_{0} + \alpha^0{}_1\alpha^0{}_0F^{10} + \alpha^0{}_0\alpha^0{}_1F^{01} + \alpha^0{}_1\alpha^0{}_1\underbrace{F^{11}}_{0} = \alpha^0{}_0\alpha^0{}_1(\underbrace{F^{10}}_{-F^{01}} + F^{01}) = 0\right)$$

$$-\frac{E'_x}{c} = F'^{01} = \alpha^0{}_\rho\alpha^1{}_\sigma F^{\rho\sigma} = \alpha^0{}_0\alpha^1{}_0\underbrace{F^{00}}_{0} + \alpha^0{}_0\alpha^1{}_1F^{01} + \alpha^0{}_1\alpha^1{}_0F^{10} + \alpha^0{}_1\alpha^1{}_1\underbrace{F^{11}}_{0} = \gamma\gamma F^{01} + (-\beta\gamma)(-\beta\gamma)\underbrace{F^{10}}_{-F^{01}} = \underbrace{\gamma^2(1-\beta^2)}_{1}F^{01} = F^{01} = -\frac{E_x}{c}$$

$$-\frac{E'_y}{c} = F'^{02} = \alpha^0{}_\rho\alpha^2{}_{\underset{2}{\sigma}} F^{\rho\sigma} = \alpha^0{}_0\alpha^2{}_2F^{02} + \alpha^0{}_1\alpha^2{}_2F^{12} = \gamma\left(-\frac{E_y}{c}\right) + (-\beta\gamma)(-B_z) = \gamma\left(-\frac{E_y}{c} + \beta B_z\right)$$

$$\boxed{E'_y = \gamma(E_y - c\beta B_z)} \qquad \boxed{E'_x = E_x}$$

$$F'^{03} = \alpha^0{}_\rho\alpha^3{}_\sigma F^{\rho\sigma} = \alpha^0{}_0\alpha^3{}_3F^{03} + \alpha^0{}_1\alpha^3{}_3F^{13} = \gamma\left(-\frac{E_z}{c}\right) + (-\beta\gamma)B_y = \gamma\left(-\frac{E_z}{c} - \beta B_y\right)$$

$$\boxed{E'_z = \gamma(E_z + c\beta B_y)}$$

$$-B'_z = F'^{12} = \alpha^1{}_\rho\alpha^2{}_\sigma F^{\rho\sigma} = \alpha^1{}_0\alpha^2{}_2F^{02} + \alpha^1{}_1\alpha^2{}_2F^{12} = (-\beta\gamma)\left(-\frac{E_y}{c}\right) + \gamma(-B_z) = \gamma\left(\frac{\beta}{c}E_y - B_z\right)$$

$$B'_y = F'^{13} = \alpha^1{}_\rho\alpha^3{}_\sigma F^{\rho\sigma} = \alpha^1{}_0\alpha^3{}_3F^{03} + \alpha^1{}_1\alpha^3{}_3F^{13} = (-\beta\gamma)\left(-\frac{E_z}{c}\right) + \gamma B_y$$

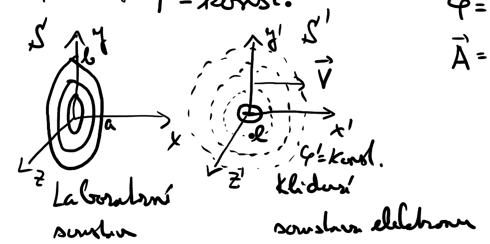
$$-B'_x = F'^{23} = \alpha^2{}_\rho\alpha^3{}_\sigma F^{\rho\sigma} = \alpha^2{}_2\alpha^3{}_3F^{23} = -B_x$$

$$\boxed{\begin{aligned}B'_z &= \gamma\left(B_z - \frac{\beta}{c}E_y\right)\\ B'_y &= \gamma\left(B_y + \frac{\beta}{c}E_z\right)\\ B'_x &= B_x\end{aligned}}$$

Pro $\vec V$ v libovolném směru

$$\vec E'_{\parallel\vec V} = \vec E_{\parallel\vec V} \qquad \vec E'_{\perp\vec V} = \gamma(\vec E_{\perp\vec V} + \vec V\times\vec B)$$
$$\vec B'_{\parallel\vec V} = \vec B_{\parallel\vec V} \qquad \vec B'_{\perp\vec V} = \gamma\left(\vec B_{\perp\vec V} - \frac{1}{c^2}\vec V\times\vec E\right)$$

57) Vypočítejte potenciály elektromagnetického pole buzeného ve vakuu bodovým nábojem e pohybujícím se (v dané inerciální soustavě S) rovnoměrně přímočaře rychlostí $\vec{V}$. Jaký tvar mají ekvipotenciální plochy $\varphi = konst.$?



$\varphi = ?$

$\vec{A} = ?$

① najdeme φ' a $\vec{A}'$ v S' $\quad \vec{A}' = 0$ (není zde mag. pole)

$$\varphi'(\vec{r}',t') = \frac{e}{4\pi\varepsilon_0}\frac{1}{r'} = \frac{e}{4\pi\varepsilon_0}\frac{1}{\sqrt{x'^2+y'^2+z'^2}} \quad \Bigg\} \quad A'^{\mu} = \begin{pmatrix} \frac{\varphi'}{c} \\ \vec{A}' \end{pmatrix} = \begin{pmatrix} \frac{\varphi'}{c} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

② spočteme $\varphi, \vec{A}$

$$A'^{\mu} = \alpha^{\mu}{}_{\nu} A^{\nu} \quad / (\alpha^{-1})^{\rho}{}_{\mu}$$

$$(\alpha^{-1})^{\rho}{}_{\mu} A'^{\mu} = \delta^{\rho}{}_{\nu} A^{\nu} = A^{\rho}$$

$$\alpha^{-1} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A^{\mu} = (\alpha^{-1})^{\mu}{}_{\nu} A'^{\nu}$$

$$\begin{pmatrix} \frac{\varphi}{c} \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\varphi'}{c} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma\frac{\varphi'}{c} \\ \beta\gamma\frac{\varphi'}{c} \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\varphi = \gamma\varphi'}$$

$$A_x = \frac{\beta\gamma\varphi'}{c} = \frac{\beta}{c}\varphi$$

$$A_y = 0$$

$$A_z = 0$$

③ Vyjádříme na původních proměnných (souřadnicích)

$$A^{\mu}(x^{\rho}) = (\alpha^{-1})^{\mu}{}_{\nu} A'^{\nu}(x'^{\sigma}) = (\alpha^{-1})^{\mu}{}_{\nu} A'^{\nu}(\alpha^{\sigma}{}_{\rho} x^{\rho})$$

$$\boxed{\varphi(\vec{r},t) = \gamma\varphi'(\vec{r}',t') = \frac{\gamma e}{4\pi\varepsilon_0}\frac{1}{\sqrt{x'^2+y'^2+z'^2}} = \frac{\gamma e}{4\pi\varepsilon_0}\frac{1}{\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}}}$$

$x' = \gamma(x-Vt) \quad y' = y \quad z' = z$

? Jak vypadají plochy $\varphi = konst.$

$$K = \varphi(\vec{r},t) = \frac{\gamma e}{4\pi\varepsilon_0}\frac{1}{\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}} \Rightarrow \gamma^2(x-Vt)^2+y^2+z^2 = \left(\frac{\gamma e}{4\pi\varepsilon_0 K}\right)^2 \Rightarrow \boxed{\frac{(x-Vt)^2}{\frac{e^2}{(4\pi\varepsilon_0 K)^2}} + \frac{y^2}{\frac{(\gamma e)^2}{(4\pi\varepsilon_0 K)^2}} + \frac{z^2}{\frac{(\gamma e)^2}{(4\pi\varepsilon_0 K)^2}} = 1}$$

Poloosy $a = \frac{e}{(4\pi\varepsilon_0 K)}$ $\quad b = \gamma a$

$(b > a)$ $\quad \gamma > 1$

Př: Hustota Lagrangeovy funkce pro elektromagnetické pole generované zdroji $\rho(\vec{r},t)$, $\vec{j}(\vec{r},t)$ je $\mathcal{L} = -\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} - j^{\mu}A_{\mu}$ najděte pohybové rovnice tohoto pole.

VZOREC $\quad \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial\mathcal{L}}{\partial q_{a,\mu}}\right) - \frac{\partial\mathcal{L}}{\partial q_a} = 0 \quad \forall a$

$\mathcal{L} = \mathcal{L}(A_{\mu}, A_{\mu,\nu}, x^{\mu})$ $\quad$ proměnné pole jsou $A_{\mu} \longleftrightarrow q_a$

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} = \frac{\partial}{\partial x^{\mu}}A_{\nu} - \frac{\partial}{\partial x^{\nu}}A_{\mu} = A_{\nu,\mu} - A_{\mu,\nu}$$

za čárkou je index podle "kterého" se derivuje

$$A_{\mu,\nu} \longleftrightarrow q_{a,\mu}$$

$$\boxed{\frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial\mathcal{L}}{\partial A_{\kappa,\lambda}}\right) - \frac{\partial\mathcal{L}}{\partial A_{\kappa}} = 0 \quad \forall\kappa = 0,1,2,3}$$

$$\frac{\partial\mathcal{L}}{\partial A_{\kappa}} = \frac{\partial}{\partial A_{\kappa}}(-j^{\mu}A_{\mu}) = -j^{\mu}\frac{\partial A_{\mu}}{\partial A_{\kappa}} = -j^{\mu}\delta^{\kappa}_{\mu} = -j^{\kappa}$$

$$\frac{\partial\mathcal{L}}{\partial A_{\kappa,\lambda}} = \frac{\partial}{\partial A_{\kappa,\lambda}}\left(-\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu}\right) = -\frac{1}{4\mu_0}\left[\left(\frac{\partial F^{\mu\nu}}{\partial A_{\kappa,\lambda}}\right)F_{\mu\nu} + F^{\mu\nu}\left(\frac{\partial F_{\mu\nu}}{\partial A_{\kappa,\lambda}}\right)\right] = -\frac{1}{4\mu_0}\left[\frac{\partial(g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma})}{\partial A_{\kappa,\lambda}}F_{\mu\nu} + F^{\mu\nu}(\delta^{\kappa}_{\nu}\delta^{\lambda}_{\mu} - \delta^{\kappa}_{\mu}\delta^{\lambda}_{\nu})\right]$$

$$= \frac{\partial}{\partial A_{\kappa,\lambda}}(A_{\nu,\mu} - A_{\mu,\nu}) = \delta^{\kappa}_{\nu}\delta^{\lambda}_{\mu} - \delta^{\kappa}_{\mu}\delta^{\lambda}_{\nu}$$

$$= -\frac{1}{4\mu_0}\left[g^{\mu\rho}g^{\nu\sigma}\frac{\partial F_{\rho\sigma}}{\partial A_{\kappa,\lambda}}F_{\mu\nu} + (F^{\lambda\kappa} - \underbrace{F^{\kappa\lambda}}_{=-F^{\lambda\kappa}})\right] = -\frac{1}{4\mu_0}\left[\underbrace{\frac{\partial F_{\rho\sigma}}{\partial A_{\kappa,\lambda}}F^{\rho\sigma}}_{=2F^{\lambda\kappa}} + 2F^{\lambda\kappa}\right] = -\frac{1}{4\mu_0}4F^{\lambda\kappa} = \frac{1}{\mu_0}F^{\kappa\lambda}$$

$$\frac{\partial}{\partial x^{\lambda}}\left(\frac{1}{\mu_0}F^{\kappa\lambda}\right) - (-j^{\kappa}) = 0 \qquad \boxed{\frac{\partial F^{\kappa\lambda}}{\partial x^{\lambda}} = -\mu_0 j^{\kappa}}$$

Kovariantní tvar Maxwellových rovnic I. série