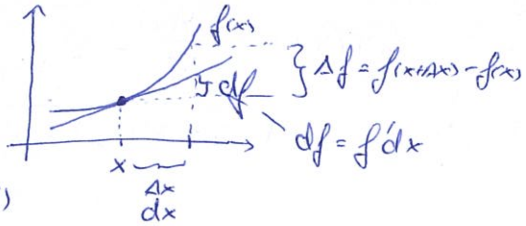
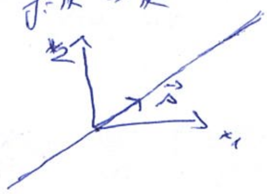
Derivace $f: \mathbb{R} \to \mathbb{R}$, $f = f(x)$ $\quad f'(x) = \left.\frac{df}{dx}\right|_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$



Derivace fce: $f: \mathbb{R}^3 \to \mathbb{R}$ ve směru $\vec{s} \in \mathbb{R}^3$ $\|\vec{s}\| = 1$ $\quad f = f(\vec{x})$

$$f_{\vec{s}}(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{s}) - f(\vec{x})}{h}$$

např. $\mathbb{R}^2$ $f: \mathbb{R}^2 \to \mathbb{R}$



Parciální derivace

$$\vec{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \frac{\partial f(\vec{x})}{\partial x_1} = f_{\vec{s}_1}(\vec{x}) = \lim_{h \to 0} \frac{f(x_1 + h\cdot 1, x_2, x_3) - f(x_1, x_2, x_3)}{h}$$

$$\vec{s}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \frac{\partial f(\vec{x})}{\partial x_2} = f_{\vec{s}_2}(\vec{x}) = \lim_{h \to 0} \frac{f(x_1, x_2 + h, x_3) - f(x_1, x_2, x_3)}{h}$$

$\vec{s}_3$

Derivace fce $f: \mathbb{R}^m \to \mathbb{R}$ $\quad f = f(\vec{y})$ je zobrazení $f': \vec{s} \to f_{\vec{s}}$ $\quad (f'(\vec{y}))\vec{s} = f_{\vec{s}}(\vec{y})$

$$f'(\vec{y}) \text{ "=" } \left(\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, \dots, \frac{\partial f}{\partial y_m} \right)$$

Řetězové pravidlo $\quad \vec{g}: \mathbb{R}^n \to \mathbb{R}^m$ $\quad \vec{g}(\vec{x}) = \vec{y}$; $\quad f: \mathbb{R}^m \to \mathbb{R}$ $\quad f(\vec{y})$ $\quad h(\vec{x}) = (f \circ \vec{g})\vec{x} = f(\vec{g}(\vec{x}))$

$$\vec{g}(\vec{x}) = \begin{pmatrix} g_1(\vec{x}) \\ \vdots \\ g_m(\vec{x}) \end{pmatrix} \qquad h: \mathbb{R}^n \to \mathbb{R}$$

$$\vec{h}'(\vec{x}) = (f \circ \vec{g})'(\vec{x}) = f'(\underbrace{\vec{g}(\vec{x})}_{\vec{y}}) \cdot \underbrace{\vec{g}'(\vec{x})} = \left(\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_m} \right) \begin{pmatrix} \frac{\partial g_1}{\partial x_1}, & \dots, & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial g_m}{\partial x_1}, & \dots, & \frac{\partial g_m}{\partial x_n} \end{pmatrix} = \left(\frac{\partial f}{\partial y_k}\frac{\partial g_k}{\partial x_1}, \dots, \frac{\partial f}{\partial y_k}\frac{\partial g_k}{\partial x_n} \right)$$

$$\boxed{\frac{\partial h(\vec{x})}{\partial x_i} = \frac{\partial (f \circ \vec{g})(\vec{x})}{\partial x_i} = \sum_{k=1}^{m} \frac{\partial f}{\partial y_k}(\vec{g}(\vec{x})) \cdot \frac{\partial g_k}{\partial x_i}(\vec{x})}$$

Př: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f = f(\vec{y}) = f(y_1, y_2) = y_1 \cdot y_2^2$ $\qquad \vec{g}: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \vec{g}(\vec{x}) = \begin{pmatrix} g_1(\vec{x}) \\ g_2(\vec{x}) \end{pmatrix} = \begin{pmatrix} x_1^2 + 3x_2 \\ x_1 \cdot x_2^2 \end{pmatrix}$

$$\frac{\partial (f \circ \vec{g})}{\partial x_1}(\vec{x}) = \frac{\partial f}{\partial y_1} \frac{\partial g_1}{\partial x_1} + \frac{\partial f}{\partial y_2} \frac{\partial g_2}{\partial x_1} = y_2^2 \cdot (2x_1) + y_1 2 y_2 (x_2^2) = (x_1 x_2^2)^2 2x_1 + (x_1^2 + 3x_2) 2 x_1 x_2^2 x_2^2 =$$

$$= 2x_1^3 x_2^4 + 2x_1^3 x_2^4 + 6x_1 x_2^5 = \underline{4x_1^3 x_2^4 + 6x_1 x_2^5}$$

$$\frac{\partial (f \circ \vec{g})(\vec{x})}{\partial x_2} = \dots$$

// ✓

$$h(\vec{x}) = (f \circ \vec{g})(\vec{x}) = y_1 \cdot y_2^2 = (x_1^2 + 3x_2)(x_1 x_2)^2 = x_1^4 x_2^4 + 3x_1^2 x_2^5 \qquad \frac{\partial h(\vec{x})}{\partial x_1} = \underline{4x_1^3 x_2^4 + 6x_1 x_2^5}$$

Ve fyzice $f: \mathbb{R}^4 \to \mathbb{R}$ $\quad f = f(x, y, z, t)$ $\quad$ diferenciál $\quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt$

explicitní závislost na čase $\left(\frac{\partial f}{\partial t} \neq 0\right)$

$$\vec{g}: \mathbb{R} \to \mathbb{R}^4 \qquad \vec{g} = \vec{g}(t) = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{t} \end{pmatrix} \qquad \begin{aligned} \tilde{x} &= vt \\ \tilde{y} &= 0 \\ \tilde{z} &= \tfrac{1}{2} g t^2 \\ \tilde{t} &= t \end{aligned} \qquad \tilde{f}(t) = f(\vec{g}(t))$$

úplná derivace podle času

$$\boxed{\dot{f} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial t}} \qquad \dot{\tilde{f}} = \frac{d\tilde{f}}{dt} = \frac{\partial f}{\partial x}\frac{d\tilde{x}}{dt} + \frac{\partial f}{\partial y}\frac{d\tilde{y}}{dt} + \frac{\partial f}{\partial z}\frac{d\tilde{z}}{dt} + \frac{\partial f}{\partial t}\underbrace{\frac{d\tilde{t}}{dt}}_{1}$$

implicitní závislost na čase

$$\dot{f} = \frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial t} \underset{\substack{\uparrow \\ \text{např.}}}{=} -2x(v) - 2y(0) - 2z\left(\tfrac{1}{2}g2t\right) + \underline{c^2 2t \cdot 1}$$

<u>Diferenciální operátory</u> (v kartézských souřadnicích)

Skalární pole je spojitá funkce $f: \mathbb{R}^3 \to \mathbb{R}$ (dostatečně hladká) proměnné x, y, z

Vektorové pole je spojité zobrazení $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ -//- nebo x_1, x_2, x_3

operátor nabla $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ $\quad (\nabla)_i = \frac{\partial}{\partial x_i}$ $\quad \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$

1) <u>gradient</u> $\vec{F} = \operatorname{grad} f = \nabla f \qquad (\nabla f)_i = \frac{\partial f}{\partial x_i}$

2) <u>divergence</u> $f = \operatorname{div} \vec{F} = \underset{\uparrow \text{s.s.s.}}{\nabla \cdot} \vec{F} \qquad \nabla \cdot \vec{F} = \frac{\partial F_i}{\partial x_i}$

3) <u>rotace</u> $\vec{G} = \operatorname{rot} \vec{F} = \underset{\uparrow \text{vektorový součin}}{\nabla \times} \vec{F} \qquad (\nabla \times \vec{F})_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k$

$\vec{a}$ ... vektorové pole

4) $(\vec{a} \cdot \nabla) = a_i \frac{\partial}{\partial x_i}$ ... skalární operátor

$(\vec{a}\nabla) f = a_i \frac{\partial f}{\partial x_i} = \vec{a} \cdot (\nabla f)$

$\Big[\underbrace{(\vec{a} \cdot \nabla)\vec{F}}_{\text{vek. pole}}\Big]_i = (\vec{a} \cdot \nabla) F_i = a_j \frac{\partial F_i}{\partial x_j}$

5) <u>Laplaceův operátor</u> $\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x_i \partial x_i}$

$\Delta f = \underset{\sum_{i=1}^{3}}{\frac{\partial^2 f}{\partial x_i^2}} \qquad \Delta \vec{F} = \begin{pmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \end{pmatrix} \qquad [\Delta \vec{F}]_i = \Delta F_i$

Př: $\underbrace{[\operatorname{rot}(\operatorname{grad} f)]_i}_{\text{vek. pole}} = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\operatorname{grad} f)_k = \varepsilon_{ijk} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right) = \underline{\varepsilon_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k}} = \varepsilon_{ijk} \frac{\partial^2 f}{\partial x_k \partial x_j} = -\varepsilon_{ikj} \frac{\partial^2 f}{\partial x_k \partial x_j} =$

$S_{jk} = S_{kj}$ symetrický

$A_{jk} = -A_{kj}$ antisymetrický

derivace jsou záměnné

$\varepsilon_{ijk} \to \varepsilon_{ikj}$: (-1)

$j\,k$: j', k' přejmenuju

$= -\underline{\varepsilon_{ij'k'} \frac{\partial^2 f}{\partial x_{j'} \partial x_{k'}}} = C$

$\boxed{A_{jk} S_{jk} = 0}$

$C = -C \Rightarrow C = 0$

Př: $\underbrace{\nabla(\vec{A} \cdot \vec{B})}_{\text{vek. pole}} = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \underline{\vec{A} \times (\nabla \times \vec{B})} + \vec{B} \times (\nabla \times \vec{A})$

$L_i = (\nabla(\vec{A} \cdot \vec{B}))_i = \frac{\partial(\vec{A} \cdot \vec{B})}{\partial x_i} = \frac{\partial(A_j B_j)}{\partial x_i} = \underline{\frac{\partial A_j}{\partial x_i} B_j} + \underline{A_j \frac{\partial B_j}{\partial x_i}}$

$P_i = (\vec{A}\nabla) B_i + (\vec{B} \cdot \nabla) A_i + \underbrace{\varepsilon_{ijk} A_j (\nabla \times \vec{B})_k}_{\sum_{jk} \varepsilon_{ijk} A_j (\nabla \times \vec{B})_k} + \varepsilon_{ijk} B_j (\nabla \times \vec{A})_k = \underbrace{A_j \frac{\partial B_i}{\partial x_j}}_{\sum_j A_j \frac{\partial B_i}{\partial x_j}} + B_j \frac{\partial A_i}{\partial x_j} + \underline{\varepsilon_{ijk} A_j \left(\varepsilon_{klm} \frac{\partial B_m}{\partial x_l}\right)} + \ldots = (*)$

$\Big\lfloor [\vec{A} \times (\nabla \times \vec{B})]_i = \varepsilon_{ijk} A_j \left(\varepsilon_{klm} \frac{\partial B_m}{\partial x_l}\right) = \varepsilon_{ijk} \underset{lmk}{\varepsilon_{klm}} A_j \frac{\partial B_m}{\partial x_l} = \varepsilon_{ijk} \varepsilon_{lmk} A_j \frac{\partial B_m}{\partial x_l} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j \frac{\partial B_m}{\partial x_l}$

$= \delta_{il} \delta_{jm} A_j \frac{\partial B_m}{\partial x_l} - \delta_{im} \delta_{jl} A_j \frac{\partial B_m}{\partial x_l} = \delta_{il} A_j \frac{\partial B_j}{\partial x_l} - \delta_{jl} A_j \frac{\partial B_i}{\partial x_l} = \underline{A_j \frac{\partial B_j}{\partial x_i} - A_j \frac{\partial B_i}{\partial x_j}}$

$(*) = \underline{\cancel{A_j \frac{\partial B_i}{\partial x_j}}} + \cancel{B_j \frac{\partial A_i}{\partial x_j}} + \underline{A_j \frac{\partial B_j}{\partial x_i}} - \underline{\cancel{A_j \frac{\partial B_i}{\partial x_j}}} + \underline{B_j \frac{\partial A_j}{\partial x_i}} - \cancel{B_j \frac{\partial A_i}{\partial x_j}}$ ✓

Pr:

$$\Delta\varphi(r) = \frac{\partial^2\varphi(r)}{\partial x_i^2} = \delta_{ij}\frac{\partial^2\varphi(r)}{\partial x_i\partial x_j} = \varphi''\frac{\delta_{ij}x_ix_j}{r^2} + \varphi'\frac{\delta_{ij}\delta_{ij}}{r} - \varphi'\frac{\delta_{ij}x_ix_j}{r^3} = \varphi''\frac{r^2}{r^2} + \varphi'\frac{3}{r} - \varphi'\frac{r^2}{r^3} = \circledast$$

$$\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad r = |\vec{r}| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{\underset{\uparrow \Sigma}{x_\ell^2}}$$

$$\delta_{ij}x_ix_j = x_ix_i = x_i^2 = r^2$$

$$\delta_{ij}\delta_{ij} = \underset{\uparrow \Sigma}{\delta_{ii}} = 3$$

$$\boxed{\frac{\partial r}{\partial x_j} = \frac{x_j}{r}}$$

$$\frac{\partial\varphi(r)}{\partial x_j} = \underbrace{\frac{d\varphi}{dr}}_{\varphi'}\cdot\frac{\partial r}{\partial x_j} = \varphi'\frac{\partial\sqrt{x_\ell^2}}{\partial x_j} = \varphi'\frac{1}{2}\underbrace{\frac{1}{\sqrt{x_\ell^2}}}_{1/r}\cdot\frac{\partial x_\ell^2}{\partial x_j} = \varphi'\frac{1}{2r}2x_\ell\overbrace{\frac{\partial x_\ell}{\partial x_j}}^{\delta_{j\ell}} = \varphi'\frac{x_\ell}{r}\delta_{j\ell} = \varphi'\frac{x_j}{r}$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial\varphi(r)}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(\varphi'\frac{x_j}{r}\right) = \frac{\partial}{\partial x_i}(\varphi')\cdot\frac{x_j}{r} + \varphi'\frac{\partial}{\partial x_i}\left(\frac{x_j}{r}\right) = \varphi''\frac{x_i}{r}\cdot\frac{x_j}{r} + \varphi'\frac{\frac{\partial x_j}{\partial x_i}r - x_j\frac{\partial r}{\partial x_i}}{r^2} =$$

$$= \varphi''\frac{x_ix_j}{r^2} + \varphi'\frac{\delta_{ij}r - x_j\frac{x_i}{r}}{r^2} = \varphi''\frac{x_ix_j}{r^2} + \varphi'\frac{\delta_{ij}}{r} - \varphi'\frac{x_ix_j}{r^3}$$

$$\boxed{\Delta\varphi(r) = \varphi'' + \frac{2}{r}\varphi'}$$