Minkowského prostoročas $\mathbb{R}^4$ - události čtyřvektor bod $(x^\mu) = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$ $\quad (x_\mu) = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} ct \\ -\vec{x} \end{pmatrix}$

$\mu = 0,1,2,3$

Pseudo - metrický tenzor

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = (g^{\mu\nu})$$

Inverzní matice k $(g_{\mu\nu})$

Interval

$$(\Delta s)^2 := c^2t^2 - x^2 - y^2 - z^2 = \underbrace{g_{\mu\nu}x^\mu}x^\nu = X_\nu x^\nu = x^\mu x_\mu \quad \leftarrow \text{nezávisí na volbě soustavy}$$

$= (ct)^2 - \vec{x}^2$ $\quad (ct, -\vec{x})\cdot\begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$ $\quad$ invariant

snižování indexů

$x_\mu = g_{\mu\nu}x^\nu$ kovariantní složky

zvedání indexů

$x^\mu = g^{\mu\nu}x_\nu$ kontravariantní složky

Postulujeme $\quad d\tau = \sqrt{1 - \frac{v^2(t)}{c^2}}\,dt = \frac{dt}{\gamma}$

vlastní čas částice

Transformace - Speciální Lorentzova

$S \quad S' \xrightarrow{} v \qquad \boxed{x'^\mu = \Lambda^\mu{}_\nu x^\nu}$ $\qquad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$x^\mu = (\Lambda^{-1})^\mu{}_\nu x'^\nu$ $\qquad (\Lambda^{-1})^\mu{}_\nu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Čtyřrychlost $\quad u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\cdot\frac{dt}{d\tau} = \gamma\begin{pmatrix} c \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \frac{c}{\sqrt{1-\frac{v^2}{c^2}}} \\ \frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix}$

invariant $\quad \underbrace{u^\mu u_\mu} = \gamma\begin{pmatrix} c \\ \vec{v} \end{pmatrix}^T\cdot\gamma\begin{pmatrix} c \\ -\vec{v} \end{pmatrix} = \gamma^2(c^2 - \vec{v}^2) = c^2$

Čtyřhybnost $\quad p^\mu = m_0 u^\mu = \begin{pmatrix} \gamma m_0 c \\ \gamma m_0 \vec{v} \end{pmatrix} = \begin{pmatrix} mc \\ m\vec{v} \end{pmatrix} = \begin{pmatrix} \frac{E}{c} \\ \vec{p} \end{pmatrix}$

klidová hmotnost

Relativistická energie

$E = mc^2 = \gamma m_0 c^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$

Relativistická hmotnost $\boxed{m = \gamma m_0}$

hybnost

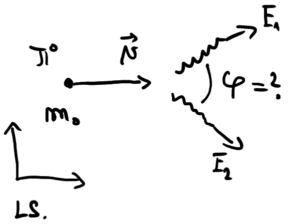
$\vec{p} = m\vec{v} = \frac{m_0\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$

Invariant $\quad p^\mu p_\mu = \begin{cases} m_0u^\mu m_0u_\mu = m_0^2c^2 \\ \begin{pmatrix} \frac{E}{c} \\ \vec{p} \end{pmatrix}^T\cdot\begin{pmatrix} \frac{E}{c} \\ -\vec{p} \end{pmatrix} = \left(\frac{E}{c}\right)^2 - \vec{p}^2 \end{cases} \Rightarrow \boxed{E^2 = p^2c^2 + m_0^2c^4}$

$E_{k} = E - E_0$

$\underline{E_0 = m_0c^2}$

47) Mezon π^0 s klidovou hmotností m_0 pohybující se rychlostí $\vec{v}$ se rozpadá na dvě stejná kvanta záření gama (fotony). Určete úhel φ který budou svírat směry pohybu fotonů.



v laboratorní soustavě

čtyřhybnost mezon $\quad p^\mu = \left(\frac{E}{c}, \vec{p}\right)$

foton $(m_0 = 0)$ $\quad E = pc \qquad \vec{p}_i = |\vec{p}_i|\vec{s}_i \quad i = 1,2$

$E = h\nu \leftarrow$ frekvence $\quad |\vec{s}_i| = 1$

Planck

fotony $\quad p_i^\mu = \left(\frac{E_i}{c}, \vec{p}_i\right) = \left(\frac{h\nu_i}{c}, \frac{h\nu_i}{c}\vec{s}_i\right) = \frac{h\nu_i}{c}(1, \vec{s}_i)$

ZZ čtyřhybnosti $\quad \boxed{p^\mu = p_1^\mu + p_2^\mu} \quad /^2 \qquad p_\mu = p_{1\mu} + p_{2\mu}$

ZZE $\quad p^0 = p_1^0 + p_2^0 \quad /\cdot c$

$E = E_1 + E_2 = h\nu_1 + h\nu_2$

stejná kvanta $\nu_1 = \nu_2$

$E = 2h\nu_1$

$$\underbrace{p^\mu p_\mu}_{\text{invariant}} = p_1^\mu p_{1\mu} + p_1^\mu p_{2\mu} + p_2^\mu p_{1\mu} + p_2^\mu p_{2\mu} = \underbrace{p_1^\mu p_{1\mu}}_{=0} + 2p_1^\mu p_{2\mu} + \underbrace{p_2^\mu p_{2\mu}}_{=0}$$

$$p_2^\mu p_{1\mu} = g^{\mu\nu}p_{2\nu}p_{1\mu} = p_{2\nu}p_1^\nu$$

$$m_0^2c^2 = 0 + 2\left(\frac{E_1}{c}, \vec{p}_1\right)\cdot\begin{pmatrix} \frac{E_2}{c} \\ -\vec{p}_2 \end{pmatrix} + 0 = 2\left(\frac{E_1E_2}{c^2} - \vec{p}_1\vec{p}_2\right) = 2\frac{h^2}{c^2}\nu_1\nu_2(1 - \underbrace{\vec{s}_1\vec{s}_2}_{\cos\varphi}) = 2\frac{h^2}{c^2}\nu_1^2(1-\cos\varphi)$$

$|\vec{s}_i| = 1$

$$m_0^2c^2 = 2\frac{h^2}{c^2}\left(\frac{E}{2h}\right)^2(1-\cos\varphi) = \frac{1}{2c^2}m_0^2\gamma^2c^4(1-\cos\varphi)$$

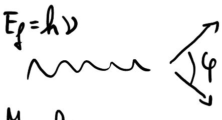
$$1 = \frac{1}{1-\frac{v^2}{c^2}}\,\frac{1}{2}(1-\cos\varphi) \iff 1 - \frac{v^2}{c^2} = \frac{1}{2}(1-\cos\varphi) = \frac{1}{2}\left(1 - \left(\cos^2\frac{\varphi}{2} - \underbrace{\sin^2\frac{\varphi}{2}}\right)\right) = \frac{1}{2}\left(1 - \cos^2\frac{\varphi}{2} + \left(1 - \cos^2\frac{\varphi}{2}\right)\right)$$

$$= \frac{1}{2}\left(2 - 2\cos^2\frac{\varphi}{2}\right) = 1 - \cos^2\frac{\varphi}{2} \Rightarrow \boxed{\cos^2\frac{\varphi}{2} = \frac{v^2}{c^2}}$$

$$\boxed{\cos\frac{\varphi}{2} = \frac{v}{c}}$$

48) Ukažte, že v nepřítomnosti vnějšího pole se foton nemůže změnit v pár elektron – pozitron.

$E_f = h\nu$



$p_f^\mu = \frac{h\nu}{c}(1,\vec{s})$

⊖ E_- $\quad p_-^\mu = \left(\frac{E_-}{c}, \vec{p}_-\right)$

⊕ E_+ $\quad p_+^\mu = \left(\frac{E_+}{c}, \vec{p}_+\right)$

$m_{0+} = m_{0-} = m_0$

ZZ EaH (p^μ) $\quad p_f^\mu = p_+^\mu + p_-^\mu \quad /^2$ $\qquad E_\pm = c\sqrt{p_\pm^2 + m_0^2c^2}$

$$\underbrace{p_f^\mu p_{f\mu}}_{0} = p_+^\mu p_{+\mu} + 2p_+^\mu p_{-\mu} + p_-^\mu p_{-\mu}$$

$$= m_0^2c^2 + 2\left(\frac{E_-}{c}\cdot\frac{E_+}{c} - \vec{p}_-\cdot\vec{p}_+\right) + m_0^2c^2 = 2\left(m_0^2c^2 + \frac{E_-E_+}{c^2} - \vec{p}_-\cdot\vec{p}_+\right) \quad /\tfrac{1}{2}$$

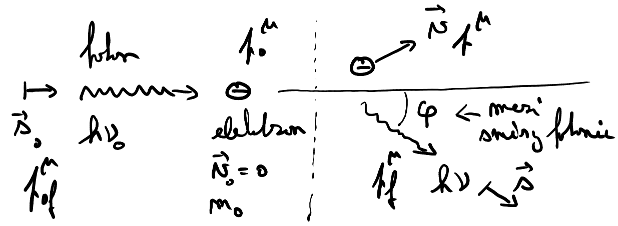
$$0 = \underbrace{m_0^2c^2}_{>0} + \underbrace{\underbrace{\sqrt{p_+^2+m_0^2c^2}}_{>p_+}\,\underbrace{\sqrt{p_-^2+m_0^2c^2}}_{>p_-}}_{>p_+p_-} - p_+p_-\underbrace{\cos\varphi}_{\le 1} > 0 \quad \text{Spor.}$$

$cp_\pm < E_\pm$

Jinak ZZE:

$$E_+ + E_- = E_f = p_f c = |\vec{p}_f|\cdot c \overset{ZZH}{=} |\vec{p}_+ + \vec{p}_-|\cdot c \le c(|\vec{p}_+| + |\vec{p}_-|) \le cp_+ + cp_- < E_+ - E_- \quad \text{Spor.}$$

49) Comptonův jev: Na elektron, který je v laboratorní soustavě v klidu dopadá foton s energií $h\nu_0$. Najděte závislost energie fotonu po srážce na úhlu φ, který svírá směr fotonu po srážce se směrem fotonu před srážkou.



$\nu(\varphi) = ?$ $\qquad E = h\nu$

v lab. soustavě $\qquad\qquad$ po srážce

fotony $\quad p_{0f}^\mu = \frac{h\nu_0}{c}(1,\vec{s}_0) \qquad p_f^\mu = \frac{h\nu}{c}(1,\vec{s})$

elektron $\quad p_0^\mu = \left(\frac{E_0}{c}, \vec{p}_0\right) = \left(\frac{m_0c^2}{c}, 0\right) \qquad p^\mu = \left(\frac{E}{c}, \vec{p}\right)$

$\vec{p}_0 = m_0\vec{v}_0 = 0$

ZZEaH $\quad \underbrace{p_{0f}^\mu + p_0^\mu} = \underbrace{p_f^\mu + p^\mu}$ $\qquad$ ZZE: $h\nu_0 + E_0 = h\nu + E$

$E = E_0 + h(\nu_0 - \nu)$

$$p_{0f}^\mu - p_f^\mu = p^\mu - p_0^\mu \quad /^2$$

$$\underbrace{p_{0f}^\mu p_{0f\mu}}_{0} - 2p_{0f}^\mu p_{f\mu} + \underbrace{p_f^\mu p_{f\mu}}_{0} = \underbrace{p^\mu p_\mu}_{m_0^2c^2} - 2p^\mu p_{0\mu} + \underbrace{p_0^\mu p_{0\mu}}_{m_0^2c^2}$$

$$-2\frac{h\nu_0}{c}(1,\vec{s}_0)\cdot\frac{h\nu}{c}\begin{pmatrix}1\\ -\vec{s}\end{pmatrix} = 2m_0^2c^2 - 2\left(\frac{E}{c},\vec{p}\right)\cdot\begin{pmatrix}\frac{E_0}{c}\\ 0\end{pmatrix} \qquad E_0 = m_0c^2$$

$$-2\frac{h^2\nu_0\nu}{c^2}(1-\vec{s}_0\vec{s}) = 2\left(m_0^2c^2 - \frac{EE_0}{c^2}\right) = 2\left(m_0^2c^2 - \frac{1}{c^2}m_0c^2\left[E_0 + h(\nu_0-\nu)\right]\right) = 2\left(\cancel{m_0^2c^2} - \cancel{m_0^2c^2} - m_0h(\nu_0-\nu)\right)$$

$$-2\frac{h^2}{c^2}\nu_0\nu(1-\cos\varphi) = 2(-m_0h(\nu_0-\nu)) \Rightarrow \frac{h}{c^2}\nu_0\nu(1-\cos\varphi) = m_0(\nu_0-\nu)$$

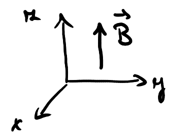
$$\nu\left(\nu_0\frac{h}{c^2}(1-\cos\varphi) + m_0\right) = m_0\nu_0$$

$$\boxed{\nu = \frac{m_0\nu_0}{m_0 + \nu_0\frac{h}{c^2}\underbrace{(1-\cos\varphi)}_{=1-\cos^2\frac{\varphi}{2}+\sin^2\frac{\varphi}{2} = 2\sin^2\frac{\varphi}{2}}} = \frac{\nu_0}{1 + \frac{2h\nu_0}{m_0c^2}\sin^2\frac{\varphi}{2}}}$$

$E(\varphi) = h\nu(\varphi)$

50) Pohyb nabité částice v magnetickém poli – cyklotronová frekvence. Určete jak se bude pohybovat nabitá částice v homogenním magnetické poli intenzity $\vec{B} = \text{konst.}$ v nerelativistickém a relativistickém případě.

souřadnice



$\vec{B} = (0,0,B)$

1) nerelativistický $\qquad$ ele. pole = 0

$$m_0\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt} = \vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) = q\vec{v}\times\vec{B}$$

$\vec{p} = m_0\vec{v}$ $\qquad$ Lorentzova

$$\frac{d\vec{v}}{dt} = \frac{q}{m_0}\vec{v}\times\vec{B}$$

$\dot{v}_x = \frac{q}{m_0}Bv_y \quad /\frac{d}{dt} \qquad \ddot{v}_x = \frac{q}{m_0}B\dot{v}_y = -\left(\frac{q}{m_0}B\right)^2 v_x$

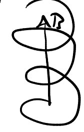
$\dot{v}_y = \frac{q}{m_0}B(-v_x)\cdot 1$

$\dot{v}_z = \frac{q}{m_0}B\cdot 0 = 0 \to v_z = v_{0z} = \text{konst} \to z = v_{0z}t + z_0$

$$\ddot{v}_x + \Big(\underbrace{\frac{q}{m_0}B}_{}\Big)^2 v_x = 0 \quad \rightarrow \quad v_x(t) = A\cos\Big(\frac{q}{m_0}Bt + \varphi_0\Big) \quad \rightarrow \quad x(t) = \frac{m_0}{qB} A\sin\Big(\underbrace{\frac{q}{m_0}}B t + \varphi_0\Big) + x_0$$

ω^2

$$v_y(t) = \frac{m_0}{qB}\dot{v}_x = -A\sin\Big(\frac{q}{m_0}Bt + \varphi_0\Big) \rightarrow y(t) = \frac{m_0}{qB} A\cos\Big(\underbrace{\frac{q}{m_0}B}_{\omega_{Ner}} t + \varphi_0\Big) + y_0$$



cyklotronová frekvence $\omega_{Ner.} = \frac{qB}{m_0}$

2, relativisticky

$$\frac{d}{dt}\left(\frac{m_0\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{d\vec{p}}{dt} = q\vec{v}\times\vec{B}$$

$$\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{E}{c^2}$$

$$\frac{d}{dt}\Big(\overbrace{\frac{E}{c^2}}^{\text{Konst}}\vec{v}\Big) = q\vec{v}\times\vec{B}$$

pro pohyb částice v Em. poli je E integrál pohybu

$\left(\frac{\partial L}{\partial t} = 0\right)$

$$\vec{v}\cdot\underbrace{\frac{d}{dt}\Big(\frac{E}{c^2}\Big)}_{=0} + \frac{E}{c^2}\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B}$$

$$\frac{d\vec{v}}{dt} = \underbrace{\frac{c^2 q}{E}}_{\text{Konst}}\vec{v}\times\vec{B}$$

stejné jako v 1

jen $\omega_{Ner} \rightarrow \omega_{Rel} = \frac{c^2 qB}{E} = \frac{qB}{m_0}\sqrt{1-\frac{v^2}{c^2}}$